Eigenvalues of Matrices

February 15, 2012

Notes

- Practice Problems:
  - 7.8 1-17 (with and without matlab)
  - 8.1 1-27 (with and without matlab)
- Homework 4 and 5 available on webwork.
- Homework 6 (eigenvalues) will be available next week.
- Test 1 – Wednesday 2/22 (Chapter 7 only)
- Monday 2/20 – Review (come with questions).
- Tuesday 2/21 – Review (7:10pm in MAGC 1.324) (come with questions).

Functions of Matrices

So here is where we are: We have these new objects called matrices, we can add them together and multiply them together provided the sizes are compatible.

We have seen now that square matrices are in some senses better behaved than rectangular matrices. We can multiply them together in any order, though the multiplication is not commutative.

Most importantly we have seen that some square matrices are invertible. We call them non-singular if they are invertible, singular if they are not invertible.

The determinant of a matrix gives a method for checking whether the matrix is singular or non-singular.

Functions of Matrices

We would like to make sense out of functions of matrices. For example consider the polynomial function

\[ p(\lambda) = \lambda^2 + 2\lambda + 1 \]

Let

\[ A = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} \]

What do we mean by \( p(A) \)?
Functions of Matrices

\[ p(A) = A^2 + 2A + 1 \]
\[ = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}^2 + 2 \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} + I \]
\[ = \begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 2 & 4 \\ -2 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]
\[ = \begin{pmatrix} 2 & 8 \\ -2 & 2 \end{pmatrix} \]

Eigenvalues of Matrices

But what do we mean by \( e^A \)?

\[ e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \ldots \]

This is a big problem to compute! We need a better way.

Diagonal Matrices

Note that in the previous question, \( e^A \), if \( A \) is a diagonal matrix, the answer is much easier. We have:

\[ A^n = \begin{pmatrix} a_{11}^n & 0 & 0 \\ 0 & a_{22}^n & 0 \\ 0 & 0 & a_{33}^n \end{pmatrix} \]

So that

\[ e^A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{3!} \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} + \ldots \]

\[ = \begin{pmatrix} e^{a_{11}} & 0 & 0 \\ 0 & e^{a_{22}} & 0 \\ 0 & 0 & e^{a_{33}} \end{pmatrix} \]
Eigenvalues and Eigenvectors

This leads to one of the most important types of questions in engineering, mathematics, and physics.

**Problem Statement**
Given a $n \times n$ matrix $A$, find a number $\lambda$ and a vector $v \neq 0$ such that

$$Av = \lambda v$$

$\lambda$ is called an **Eigenvalue**

$v$ is called an **Eigenvector** associated to $\lambda$. Sometimes also called an **Eigendirection**.

The set of all eigenvalues is called the **spectrum** of $A$.

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### Eigenspaces

Given a matrix $A$ and one of its eigenvalues $\lambda$ the set of eigenvectors of $A$ associated to $\lambda$ gives a subspace of $\mathbb{R}^n$.

This is related to our previous question because, in the eigenspaces $A$ acts like a diagonal matrix.

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### Finding Eigenvalues

Note that an eigenvalue $\lambda$ of $A$ satisfies the equation

$$Av = \lambda v$$

for some non-zero $v$.

In other words, there is a non-trivial solution to the homogeneous linear equation

$$Av - \lambda v = (A - \lambda I)v = 0$$

So we must have

$$D(\lambda) = \det(A - \lambda I) = 0$$

$D(\lambda)$ is called the **characteristic polynomial** of $A$.
Note that for $A_{n \times n}$

$$D(\lambda) = \det(A - \lambda I)$$

is a polynomial of degree $n$ in $\lambda$.

**Theorem**

The eigenvalues of a $n \times n$ matrix $A$ are the roots of the characteristic polynomial of $A$. Including multiplicity, there are $n$ eigenvalues (some may be complex).

**Example: $2 \times 2$**

Consider

$$A = \begin{pmatrix} -5 & 2 \\ 2 & -2 \end{pmatrix}$$

The characteristic polynomial is

$$D(\lambda) = \begin{vmatrix} -5 - \lambda & 2 \\ 2 & -2 - \lambda \end{vmatrix}$$

Solutions are given by

$$\lambda_{1,2} = \frac{-7 \pm \sqrt{49 - 24}}{2} = \frac{-7 \pm \sqrt{25}}{2} = -1, -6$$
Example: $2 \times 2$

Now to find the eigenvectors we solve the equations for $x_1$ and $x_2$:

\[
\begin{pmatrix}
-5 - (-1) & 2 \\
2 & -2 - (-1)
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

Using $rref$ we find

\[
\begin{pmatrix}
1 & -\frac{1}{2} \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

and from here we see that a non-trivial solution is given by

\[
x_2 \left( \frac{1}{2} \right) \quad \text{and taking} \quad x_2 = 2 \quad \text{we have} \quad v_1 = \begin{pmatrix}
\frac{1}{2}
\end{pmatrix}
\]
Now to find the eigenvectors we solve the equations for $x_1$ and $x_2$:

\[
\begin{pmatrix}
-5 - (-6) & 2 \\
2 & -2 - (-6)
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} =
\begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

Using Gauss-Jordan Elimination (by hand) we find

\[
\begin{pmatrix}
1 & 2 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} =
\begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

and from here we see that a non-trivial solution is given by $x_2 \begin{pmatrix}
-2 \\
1
\end{pmatrix}$ and taking $x_2 = 1$ we get $v_2 = \begin{pmatrix}
-2 \\
1
\end{pmatrix}$.
We find 
\[ v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \]
is associated with \( \lambda = -1 \) and 
\[ v_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \]
is associated with \( \lambda = -6 \).

Example: Complex Eigenvalues

As the eigenvalues are fundamentally roots of a polynomial equation we are not surprised that complex eigenvalues are possible and likely.

Consider 
\[ A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \]

Example: 3 × 3

Consider the matrix 
\[ A = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix} \]

Multiplicity

We find in the previous example that the eigenvalues are 
\( \lambda_1 = 5, \lambda_2 = \lambda_3 = -3 \) with respective eigenvectors 
\[ v_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, v_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \]

We say that \(-3\) is a \text{multiple} eigenvalue of \text{algebraic multiplicity} 2, because it is a solution of the characteristic polynomial of multiplicity 2.

We say that it also has \text{geometric multiplicity} 2, because it has two eigenvectors associated to it.
Defect of an Eigenvalue

The \textit{Defect} of an eigenvalue is the difference between its algebraic multiplicity and geometric multiplicity:

\[ \Delta \lambda = M_\lambda - m_\lambda \]

Note that the geometric multiplicity is always at most the algebraic multiplicity.

8.2 Example: Application of Eigenvalues

Markov Process: A Markov process is a dynamical system which evolves by matrix multiplications.

- Every year 3 percent of the population of Dallas moves to Houston, and 1 percent to the RGV.
- Every year 2.8 percent of the population of Houston moves to Dallas, and 2 percent to the RGV.
- Every year 1 percent of the population of the RGV moves to Dallas and Houston respectively.
- Currently there are 1.3 million in Dallas, 2.3 million in Houston, 1.1 million in RGV.

The population now is given by the vector

\[ v_0 = \begin{pmatrix} 1.3 \\ 2.3 \\ 1.1 \end{pmatrix} \]

The population in year \( n \) is given by the matrix product:

\[ v_n = A v_{n-1}, \quad \text{where} \quad A = \begin{pmatrix} 0.96 & 0.028 & 0.01 \\ 0.03 & 0.952 & 0.01 \\ 0.01 & 0.02 & 0.98 \end{pmatrix} \]

Note that the sum of each column is 1. Why?

So we have

\[ v_n = A v_{n-1} = A^2 v_{n-2} = \cdots = A^n v_0 \]

Now suppose we look for the eigenvalues of \( A \), using \([u, \lambda] = \text{eig}(A)\) for example.

I find: 0.92661, 0.96539, 1. With eigenvectors

\[ u_1 = \begin{pmatrix} 0.59994 \\ -0.77961 \\ 0.17967 \end{pmatrix}, \quad u_2 = \begin{pmatrix} -0.43970 \\ -0.37596 \\ 0.81567 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0.50848 \\ 0.46833 \\ 0.72257 \end{pmatrix} \]

By definition we have that

\[ A^* u_1 = 0.92661 u_1 \] etc.
8.2 Example: Application of Eigenvalues

Notice that we can write \( \mathbf{v}_0 \) in terms of these vectors. Why? How?

\[
\mathbf{v}_0 = \begin{bmatrix} 1.3 \\ 2.3 \\ 1.1 \end{bmatrix} = -0.84762\mathbf{u}_1 - 0.91475\mathbf{u}_2 + 2.76570\mathbf{u}_3
\]

Then multiplying by \( A^n \) we find:

\[
A^n\mathbf{v}_0 = -0.84762\lambda_1^n\mathbf{u}_1 - 0.91475\lambda_2^n\mathbf{u}_2 + 2.76570\lambda_3^n\mathbf{u}_3
\]

Note that \( \lambda_1 \) and \( \lambda_2 < 1 \), so positive powers of them are smaller and smaller. Thus over the long term (i.e. \( n \to \infty \)) only the third term in this sum survives.

\[
\mathbf{v}_n = A^n\mathbf{v}_0 \approx 2.76570\mathbf{u}_3
\]
Consider the matrix
\[
\begin{pmatrix}
3.0 & 0.4 \\
0.4 & 3.0
\end{pmatrix}
\]
As a transformation of the plane. It takes a point \((x_1, x_2)\) to
\[
\begin{pmatrix}
y_1 \\
y_2
\end{pmatrix} = \begin{pmatrix}
3.0 & 0.4 \\
0.4 & 3.0
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\]
For example the point \((1, 2)\) is taken to \((3.8, 6.4)\).

The characteristic polynomial is
\[
D(\lambda) = (3 - \lambda)^2 - 0.16 = \lambda^2 - 6\lambda + 8.84
\]
with solutions \(\lambda_1 = 2.6\), \(\lambda_2 = 3.4\)

The eigenvectors are respectively:
\[
v_1 = \begin{pmatrix}
-1 \\
1
\end{pmatrix} \text{ and } v_2 = \begin{pmatrix}
1 \\
1
\end{pmatrix}
\]

We conclude that \(A\) acts on vectors \((-1, 1)^T\) by scaling it by 2.4 and on vectors \((1, 1)^T\) by scaling them by 3.4.
Finally we note that \((-1, 1)\) and \((1, 1)\) give a basis of \(\mathbb{R}^2\).

So every \((x_1, x_2)\) can be written as a linear combination of these vectors.

A circle centered at the origin is deformed into an ellipse with a major axis along the vector \((1, 1)\) and a minor axis along \((-1, 1)\).

**Example: Deformations of the plane**

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**A acts like a diagonal matrix on eigenvectors**

We began this section by consider the question of what we can say about \(e^A\).

We have seen that for a diagonal matrix

\[ e^A = \text{diag}[e^{a_{11}}, e^{a_{22}}, \ldots, e^{a_{nn}}] \]

Suppose \(A\) is not diagonal; and \(\lambda\) and \(v_\lambda\) is a eigenpair. Then

\[ Av_\lambda = \lambda v_\lambda \]
A acts like a diagonal matrix on eigenvectors

- We began this section by consider the question of what we can say about $e^A$.
- We have seen that for a diagonal matrix
  \[ e^A = \text{diag} \left[ e^{a_{11}}, e^{a_{22}}, \ldots, e^{a_{nn}} \right] \]
- Suppose $A$ is not diagonal; and $\lambda$ and $v_\lambda$ is a eigenpair. Then
  \[ A v_\lambda = \lambda v_\lambda \]
  \[ A^n v_\lambda = \lambda^n v_\lambda \]

More interestingly we see that

\[ e^A v_\lambda = e^{\lambda} v_\lambda \]

Eigenvalues of Matrices

Spectral Theorem

**Theorem (Spectral Theorem)**

*More generally we will find that, if $f(x)$ is a function with a “nice enough” Taylor series we will have*

\[ f(A) v_\lambda = f(\lambda) v_\lambda \]